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ON THE NUMBER OF TERMS IN THE MIDDLE OF ALMOST SPLIT SEQUENCES OVER TAME ALGEBRAS

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ABSTRACT. Let A be a finite dimensional tame algebra over an algebraically closed field k. It has been conjectured that any almost split sequence $0 \to X \to \oplus_{i=1}^{n} Y_i \to Z \to 0$ with Y_i indecomposable modules has $n \leq 5$ and in case $n = 5$, then exactly one of the Y_i is a projective-injective module. In this work we show this conjecture in case all the Y_i are directing modules, that is, there are no cycles of non-zero, non-iso maps $Y_i = M_1 \to M_2 \to \cdots \to M_s = Y_i$ between indecomposable A-modules. In case, Y_1 and Y_2 are isomorphic, we show that $n \leq 3$ and give precise information on the structure of A.

Let A be a finite dimensional algebra over an algebraically closed field k. We denote by mod_A the category of finite dimensional left A-modules (an object in mod_A is simply called a module).

For a non-projective indecomposable module X, there exist an indecomposable non-injective module $\tau_A X$ called the Auslander-Reiten translate and an almost split sequence $0 \to \tau_A X \to E \to X \to 0$ (see [2], [18]). Since their introduction, almost split sequences have played a central role in the representation theory of algebras (see for example [2]).

For an almost split sequence $0 \to \tau_A X \to E \to X \to 0$, consider the indecomposable decomposition $E = \bigoplus_{i=1}^{s(X)} Y_i$. There has been considerable attention paid to the relation between properties of the algebra A and the values $s(X)$ for different modules X (and of course between properties of X and the value $s(X)$). Among other interesting results we recall that if A is representation finite, then $s(X) \leq 4$, for every indecomposable non-projective module X [3], [7] (see also [10] and [12]).

It has been conjectured by S. Brenner that, for A a tame algebra, $s(X) \leq 5$ for every indecomposable non-projective module X. This is known to hold for many examples, in particular for the important case of hereditary tame algebras.

To state the main results of this work, we recall some concepts. A *cycle* in mod_A is a sequence $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} X_s = X_0$ of non-zero non-isomorphism maps between indecomposable modules; the cycle is said to be *finite* if $f_i \notin \operatorname{rad}_A^\infty(X_{i-1}, X_i)$, for all $i = 1, \ldots, s$. An indecomposable module X is said to be *directing* if it does not belong to a cycle in mod_A. The algebra A is said to be *cycle-finite* if all cycles in mod_A are finite. We recall that a cycle finite algebra is tame [1].

The *Coxeter matrix* ϕ_A of A and its *spectral radius* $\rho(\phi_A) = \max\{\|\lambda\| \colon \lambda \text{ eigenvalue of } \phi_A\}$ are important invariants (see for example [5], [14], [17], [20]). In case

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A is a triangular algebra, we denote by

$$\tilde{\rho}(A) = \max\{\rho(\phi_B) \colon B = A/AeA \text{ for some idempotent } e \in A\}.$$

Theorem 1. *Assume that A is a triangular algebra such that*

$$\tilde{\rho}(A) \leq \frac{t-2+\sqrt{t^2-4t}}{2}$$

for some natural number $t \geq 4$. Let $0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ be an almost split sequence in mod_A such that Y_i is directing $(i = 1, \dots, s(X))$. Then $s(X) \leq t+1$.

Theorem 2. *Assume A is a tame algebra and let $0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ be an almost split sequence in mod_A such that Y_i is directing $(i = 1, \dots, s(X))$.*

Then the following holds:

(a) *$s(X) \leq 5$. Moreover, if $s(X) = 5$, then for some $j \in \{1, \dots, s(X)\}$, the module Y_j is projective and injective.*

(b) *If $s(X) \geq 3$, and $Y_1 \cong Y_2$, then $s(X) = 3$ and the module Y_3 is projective and injective.*

Theorem 3. *Assume A is a cycle-finite algebra. Then for any almost split sequence $0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ we have $s(X) \leq 5$. If $s(X) = 5$, there is some Y_j $(1 \leq j \leq s)$ which is projective and injective.*

We prove the theorems in section 2 after some preliminary considerations. We gratefully acknowledge support of CONACYT and DGAPA, UNAM.

1. Cycles and almost split sequences

1.1. Let $H = k\Delta$ be the path algebra of a quiver Δ without oriented cycles (see [6]). A *tilting module* T in mod_H satisfies: $\mathrm{Ext}^1_H(T,T) = 0$ and there is an exact sequence $0 \to H \to T' \to T'' \to 0$ with $T', T'' \in \mathrm{add}\, T$ (see [18]). For a tilting module ${}_HT$, the algebra $B = \mathrm{End}_H(T)$ is called a *tilted algebra* of type Δ.

We recall that an indecomposable B-module X is *sincere* if $\mathrm{Hom}_B(P,X) \neq 0$ for every projective B-module P. If X is a sincere directing indecomposable B-module, then B is a tilted algebra [18, p. 375].

1.2. The *Auslander-Reiten quiver* Γ_A of A has as vertices representatives of the iso-classes of indecomposable modules; there are as many arrows $X \to Y$ in Γ_A as $\dim_k \mathrm{rad}_A(X,Y)/\mathrm{rad}^2_A(X,Y)$. A path $Y_0 \to Y_1 \to \cdots \to Y_s$ in Γ_A is said to be *sectional* if $\tau_A Y_{i+1} \neq Y_{i-1}$ for all $i = 1, \dots, s-1$. In case $Y_0 = Y_s$, we have a *sectional cycle* if $\tau_A Y_{i+1} \neq Y_{i-1}$ for all $i = 1, \dots, s \bmod s$. A sectional path in Γ_A contains no sectional cycle [4].

A component $\mathcal{C}$ of Γ_A is *directing* if it is formed by directing modules.

1.3. Let $\eta \colon 0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ be an almost split sequence in mod_A and let $B(\eta)$ be a quotient A/AeA, with e idempotent in A, of minimal dimension such that η is formed by $B(\eta)$-modules. By [18], one of the modules $\tau_A X$, X or Y_i $(1 \leq i \leq s(X))$ is sincere as $B(\eta)$-module.

Lemma. *Let* $\eta\colon 0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ *be an almost split sequence in* mod_A *such that* Y_i *is directing as* $B(\eta)$*-module, for* $i = 1, \dots, s(X)$*. Then*

(a) X *and* $\tau_A X$ *are directing as* $B(\eta)$*-modules;*

(b) $B(\eta)$ *is a tilted algebra.*

Proof. Consider the almost split sequence $\eta\colon 0 \to \tau_A X \xrightarrow{(h_i)} \bigoplus_{i=1}^{s(X)} Y_i \xrightarrow{(g_i)} X \to 0$. Let $X = Z_0 \xrightarrow{f_1} Z_1 \to \cdots \to Z_{s-1} \xrightarrow{f_s} Z_s = X$ be a cycle in $\mathrm{mod}_{B(\eta)}$. The map $f_s\colon Z_{s-1} \to Z_s = X$ factorizes through the sink map $(g_i)_i\colon \bigoplus_{i=1}^{s(X)} Y_i \to X$ and for some $j \in \{1, \dots, s(X)\}$ we get a non-zero map $f'_s\colon Z_{s-1} \to Y_j$. Therefore we get a cycle

$$Y_j \xrightarrow{g_j} X = Z_0 \xrightarrow{f_1} Z_1 \to \cdots \to Z_{s-1} \xrightarrow{f'_s} Y_j.$$

A contradiction. Hence X (and similarly $\tau_A X$) is directing. The algebra $B(\eta)$ is tilted by (1.1). □

1.4 Proposition. *Let* X *be a directing indecomposable non-projective module and* $\eta\colon 0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ *be the corresponding almost split sequence. Assume* $s(X) \geq 3$*. Then one of the following conditions holds:*

(a) $B(\eta)$ *is a tilted algebra and* X *belongs to a directing component of* $\Gamma_{B(\eta)}$*;*

(b) *there exists a sectional path* $\tau_A X = Z_0 \to Z_1 \to \cdots \to Z_s$ *in* $\Gamma_{B(\eta)}$ *with* Z_s *an injective module.*

Proof. We shall denote $B = B(\eta)$. Assume first that Y_i is directing as B-module for $i = 1, \dots, s(X)$. By (1.3), B is a tilted algebra. By [9], we know the structure of Γ_B: there is a postprojective, a preinjective and a connecting component (some of these components may coincide) and components of type $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/(n)$, possibly with inserted ray modules or coinserted coray modules. Since $s(X) \geq 3$, then X belongs to a postprojective, preinjective or connecting component of Γ_B, all of which are directing. Hence (a) holds.

Assume Y_1 is not directing. Let $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \to \cdots \xrightarrow{f_s} Z_s = Y_1$ be a cycle in mod_B. By (1.2), we may assume that one of the following situations occurs:

(1) $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \to \cdots \xrightarrow{f_r} Z_r \xrightarrow{f_{r+1}} Z_{r+1} \to \cdots \to Y_1$ such that $Z_0 \xrightarrow{f_1} Z_1 \to \cdots \xrightarrow{f_r} Z_r$ is a sectional path in Γ_B and $\tau_B Z_{r+1} = Z_{r-1}$;

(2) for all $i \in \mathbb{N}$, there is a sectional path in Γ_B, $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \to \cdots \xrightarrow{f_i} Z_i$ and a path $Z_i \to Z'_{i+1} \to \cdots \to Y_1$ in mod_B.

If some Z_i is injective ($1 \leq i \leq r$ in case (1) or $1 \leq i$ in case (2)), then (b) holds. We assume that no Z_i is injective in order to get a contradiction.

First observe that situation (1) cannot happen. Otherwise, we get a cycle

$$Y_1 \xrightarrow{h_1} X \to \tau_B^- Y_1 \to \tau_B^- Z_1 \to \cdots \tau_B^- Z_{r-1} = Z_{r+1} \to \cdots \to Y_1 \xrightarrow{h_1} X,$$

where $h_1\colon Y_1 \to X$ is an irreducible map. Contradicting that X is a directing A-module.

Let n be the number of iso-classes of simple A-modules and consider a sectional path $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2 \to \cdots \xrightarrow{f_n} Z_n$ in Γ_B as given in situation (2). Moreover, there is a path $Z_n \to Z'_{n+1} \to \cdots \to Y_1$ in mod_B. We shall prove that $\bigoplus_{i=0}^{n} Z_i$ is a partial cotilting module, which yields the desired contradiction. Indeed, since the number of summands of the partial cotilting module is not bigger than n, then we get $Z_i \cong Z_j$ for some $j > i$. By (1.2), the cycle $Z_i \to \cdots Z_{j-1} \to Z_j$ is not sectional and hence $Z_{j-1} \cong \tau_B Z_{i+1}$ or $Z_{j-2} \cong \tau_B Z_i$ which yields a cycle through X as above.

Let us first show that $i\dim_B Z_i \le 1$, $i = 0, \dots, n$. Otherwise $i\dim_B Z_i > 1$ and there are an indecomposable projective B-module P and a map $0 \ne g \in \text{Hom}_B(\tau_B^- Z_i, P)$ (see [18]). Since $\bigoplus_{i=1}^{s(X)} Y_i$ is a sincere B-module, there are some $j \in \{1, \dots, s(X)\}$ and a map $0 \ne g' \in \text{Hom}_B(P, Y_j)$. We get a cycle

$$X \to \tau_B^- Y_j \to \cdots \to \tau_B^- Z_i \xrightarrow{g} P \xrightarrow{g'} Y_j \xrightarrow{h_j} X,$$

again a contradiction.

Let $i, j \in \{0, \dots, n\}$; we show that $\text{Ext}_B^1(Z_i, Z_j) = 0$. Otherwise, there is a map $0 \ne g \in \text{Hom}_B(\tau_B^- Z_j, Z_i)$. Then we get a cycle

$$\begin{aligned} X \to \tau_B^- Y_j \to \cdots \to \tau_B^- Z_j \xrightarrow{g} Z_i \xrightarrow{f_{i+1}} Z_{i+1} \to \cdots \\ \cdots \to Z_n \to \cdots \to Y_1 \xrightarrow{h_i} X, \end{aligned}$$

and a contradiction. This shows that $\bigoplus_{i=0}^{n} Z_i$ is a partial cotilting module, which completes the proof. □

1.5. We say that X is a *predecessor* of Y in Γ_A (and Y a *successor* of X) if there is a path $X = Z_0 \to Z_1 \to \cdots \to Z_s = Y$ in Γ_A.

Proposition. *Let* $\eta\colon 0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ *be an almost split sequence such that* $Y_1 \cong Y_2$. *Then*

(a) *If* $\dim_k \tau_A X < \dim_k Y_1$, *then* $\tau_A^{-n} X$ *and* $\tau_A^{-n} Y_i$ $(1 \le i \le s(X))$ *are well defined for all* $n \ge 0$. *Moreover,* X *has no injective successors and does not belong to any oriented cycle in* Γ_A.

(b) *If* $\dim_k \tau_A X > \dim_k Y_1$, $\tau_A^n X$ *and* $\tau_A^n Y_1$ *are well defined for all* $n \ge 0$. *Moreover,* $\tau_A X$ *has no projective predecessors and does not belong to any oriented cycle in* Γ_A.

Proof. (a): Let $\eta\colon 0 \to \tau_A X \xrightarrow{(h_i)} \bigoplus_{i=1}^{s(X)} Y_i \xrightarrow{(g_i)} X \to 0$ be an almost split sequence with $Y_1 \cong Y_2$ and assume that h_1 is a monomorphism. Then for all $1 \le i \le s(X)$, $\dim_k Y_i < \dim_k X$. In particular, no Y_i is injective. Since the almost split sequence starting at Y_1 has X at least twice as a summand of its middle term, then the above argument implies that X is not injective. Moreover, $\dim_k \tau_A X < \dim_k X < \dim_k \tau_A^- X$. We proceed in this way to show that $\tau_A^{-n} X$ and $\tau_A^{-n} Y_i$ $(1 \le i \le s(X))$ are well defined for all $n \ge 0$.

Assume that Y is a minimal injective successor of X. Then we find a sectional path $\tau_A^s X = Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} Z_t = Y$, for some $s \geq 0$. Since there is an irreducible monomorphism $Z_1 \to \tau_A^- Z_0$, then Z_1 is not injective and there is an irreducible monomorphism $Z_2 \to \tau_A^- Z_1$. Continuing this way, we find a monomorphism $Z_t \to \tau_A^- Z_{t-1}$, which is a contradiction.

Let Γ be the right stable component of Γ_A containing X. Then Γ contains all the successors of X and it has no periodic modules (since $\dim_k \tau_A^{-n} X$ grows with $n \geq 0$ and hence X is not periodic). If X belongs to an oriented cycle in Γ_A, then the cycle belongs to Γ and by [10], Γ has trivial valuation, a contradiction.

(b): With the notation above, assume that h_1 is an epimorphism. In particular, Y_1 is not projective. Since the almost split sequence ending at Y_1 has $\tau_A X$ at least twice as a summand of its middle term, then $\dim_k \tau_A X < \dim_k \tau_A Y_1$ and $\tau_A X$ is not projective. By (a), $\dim_k \tau_A^2 X > \dim_k \tau_A Y_1$. We proceed by induction to show that $\tau_A^n X$ and $\tau_A^n Y_1$ are well defined for all $n \geq 0$.

If, for some $3 \leq j \leq s(X)$, the module Y_j is not projective, then $\dim_k \tau_A^2 X > \dim_k \tau_A Y_j$. By induction, $\tau_A^n Y_j$ is well defined. The second part of the statement follows from dual arguments to those used in (a). □

1.6. The following results of S. Liu are important in the proof of our theorems.

Proposition [9]. *Let* $0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ *be an almost split sequence in* mod_A. *Then*

(a) *Assume there is a sectional path* $\tau_A X = Z_0 \to Z_1 \to \cdots \to Z_t$ *in* Γ_A *with* Z_t *injective. Assume moreover that* $s(X) > 4$ *or that* $s(X) = 4$ *but no* Y_i *is injective* ($i = 1, \ldots, s(X)$). *Then* $\tau_A X$ *has no projective predecessor in* Γ_A *and* $\tau_A X$ *does not belong to a finite cycle in* mod_A.

(b) *If* $\tau_A X$ *belongs to a finite cycle in* mod_A, *then* $s(X) \leq 4$ *and in case* $s(X) = 4$, *there is a* Y_i ($1 \leq i \leq s(X)$), *which is projective.*

1.7. The following result generalizes [3] quoted in the introduction.

Theorem [9]. *Let* $\eta\colon 0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ *be an almost split sequence such that* $\tau_A X$ *has a projective predecessor and* X *has an injective successor in* Γ_A. *Then* $s(X) \leq 4$ *and in case* $s(X) = 4$, *there is a* Y_i ($1 \leq i \leq 4$) *which is projective and injective.*

2. Proof of the main results

2.1. *Proof of the Theorem 1.* Let $0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ be an almost split sequence in mod_A and assume that Y_i is directing ($1 \leq i \leq s(X)$). Assume that $\tilde{\rho}(A) \leq \frac{t-1+\sqrt{t^2-4t}}{2}$ for a natural number $t \geq 4$. We assume that $s = s(X) \geq 5$.

Let $B = B(\eta)$. By the proof of (1.4), we may assume that B is a tilted algebra and X belongs to a directing component $\mathcal{C}$ of Γ_B.

By (1.7), we may assume that X has no injective successors in $\mathcal{C}$. Suppose that Y_s is injective in mod_B. Since $\tau_A X \to Y_s$ is a sectional map, by (1.6), we get that $\tau_A X$ has no projective predecessors in $\mathcal{C}$. Moreover, no Y_i, $1 \leq i \leq s-1$, is projective. Indeed, if Y_1 were projective, then

$$\dim_k Y_1 + \dim_k Y_s > \dim_k \tau_{A} X + \dim_k X = \sum_{i=1}^{s} \dim_k Y_i,$$

a contradiction. By duality, we may assume that one of the following situations holds:

(1) $0 \to \tau_{B}^{m+1} X \to \bigoplus_{i=1}^{s} \tau_{B}^{m} Y_i \to \tau_{B}^{m} X \to 0$ is a well-defined almost split sequence in mod_B, for all $m \geq 0$;

(2) Y_s is projective and injective and $0 \to \tau_{B}^{m+1} X \to \bigoplus_{i=1}^{s-1} \tau_{B}^{m} Y_i \to \tau_{B}^{m} X \to 0$ is a well-defined almost split sequence in mod_B, for all $m \geq 1$.

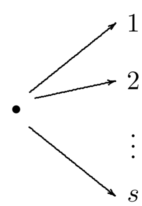
In case (1) we shall prove that $s \leq t$; in case (2), the same proof yields $s-1 \leq t$ and hence the result.

Assume that Y_s is not projective (i.e. case (1)). Consider the Grothendieck group $K_0(B)$ and the Coxeter matrix ϕ_B as a linear transformation $\phi_B \colon K_0(B) \to K_0(B)$. Since B is tilted, there are a quiver Δ and a tilting $k\Delta$-module T such that $B = \mathrm{End}_{k\Delta}(T)$. Therefore, there is an isometry $\sigma \colon K_0(k\Delta) \to K_0(B)$, $[L] \mapsto [\mathrm{Hom}_{k\Delta}(T, L)] - [\mathrm{Ext}^1_{k\Delta}(T, L)]$, where $[Y]$ denotes the class in $K_0(B)$ of a B-module Y. Moreover, $\phi_B \sigma = \sigma \phi_\Delta$ (see [18]). In particular, $\rho(\phi_B) = \rho(\phi_\Delta)$.

Observe that Δ contains the subquiver $\Delta(s_1, \dots, s_m)$ formed by one source 0, sinks $1, \dots, m$ and s_i arrows from 0 to i such that $\sum_{i=1}^{m} s_i = s$. Indeed, $\Delta = \mathcal{S}^{op}$, where $\mathcal{S}$ is the slice in the component $\mathcal{C}$ formed as the full subquiver of $\mathcal{C}$ whose vertices Z are the starting points of sectional paths to X. Since $\mathcal{S}$ contains the arrows $Y_i \to X$, $i = 1, \dots, s$, we get the claim (of course, m is the number of isoclasses among the Y_i). Moreover, by [20], we have $\rho(\phi_\Delta) \geq \rho(\phi_{\Delta(s_1,\dots,s_m)})$.

On the other hand, $\rho(\phi_{\Delta(s_1,\dots,s_m)}) = \frac{b-2+\sqrt{b^2-4b}}{2}$, where $b = \sum_{i=1}^{m} s_i^2$.

Hence $\rho(\phi_{\Delta(s_1,\dots,s_m)}) \geq \frac{s-2+\sqrt{s^2-4s}}{2} = \rho(\phi_{\Delta_0})$, where

$\Delta_0\colon$ 

Altogether we get

$$\frac{t-2+\sqrt{t^2-4t}}{2} \geq \tilde{\rho}(A) \geq \rho(\phi_B) = \rho(\phi_\Delta) \geq \rho(\phi_{\Delta_0}) = \frac{s-2+\sqrt{s^2-4s}}{2}.$$

Hence $t \geq s$, as desired. □

2.2. We recall that the algebra A is said to be *tame* if for every $d \in \mathbb{N}$ there are finitely many $A - k[t]$-bimodules $M_1, \dots, M_s$ which are finitely generated free as right $k[t]$-modules and such that every indecomposable A-module X with $\dim_k X = d$ is isomorphic to $M_i \otimes_{k[t]} S$ for some $1 \leq i \leq s$ and some simple $k[t]$-module S.

From now on we will assume that A is a basic connected algebra of the form $A = kQ/I$ (see [6]). The following result follows an idea of von Höhne (see [12]).

Proposition. *Let $A = kQ/I$ be a tame algebra and let X be a non-projective indecomposable module such that the almost split sequence $0 \to \tau_A X \to \bigoplus_{i=1}^{s} Y_i \to X \to 0$ has directing middle terms Y_i, $i = 1, \dots, s$. Then for all vertices $i \in Q_0$ we have*

$$|\dim_k \tau_A X(i) - \dim_k X(i)| \leq 2.$$

Proof. Let $B = B(\eta)$. By (1.3), B is a tame tilted algebra. Then it easily follows that $p\dim_B X \leq 1$ and $\mathrm{Hom}_B(X, {}_BB) = 0$. Therefore, by [18], $[\tau_A X] = [\tau_B X] = \phi_B[X]$.

Consider the Euler (non-symmetric) bilinear form $\langle -, - \rangle \colon K_0(B) \times K_0(B) \to \mathbb{Z}$ defined by $\langle [L], [M] \rangle = \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{Ext}_B^i(L, M)$ and the corresponding quadratic form χ_B. Since $\mathrm{g}\ell \dim B \leq 2$, then χ_B is also the Tits form of B and therefore χ_B is weakly non-negative (see for example [12]). We have $\chi_B([X]) = 1 = \chi_B([P_i])$, for the indecomposable projective module P_i corresponding to the vertex $i \in Q_0$. Then

$$\begin{aligned} 0 \leq \chi_B([X] + [P_i]) &= 2 + \langle [P_i], [X] \rangle + \langle [X], [P_i] \rangle \\ &= 2 + \dim_k X(i) - \langle [P_i], \phi_B[X] \rangle \\ &= 2 + \dim_k X(i) - \dim_k \tau_A X(i). \end{aligned}$$

Hence, $\dim_k \tau_A X(i) - \dim_k X(i) \leq 2$ and similarly we get the other inequality. □

Corollary. *Let A be a tame algebra and $\mathcal{C}$ be a directing component of Γ_A. Let $X \in \mathcal{C}$ be such that $\tau_A^m X$ is well-defined for all $m \geq 0$. Then $\lim_{m\to\infty} \sqrt[m]{\dim_k \tau_A^m X} = 1$.*

2.3. *Proof of the Theorem 2.* Let A be a tame algebra and consider an almost split sequence $\eta \colon 0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ such that Y_i is directing, $1 \leq i \leq s(X)$.

(a): Assume that $s = s(X) \geq 5$.

Let $B = B(\eta)$ be a tilted algebra by (1.4) and assume that X belongs to a directing component $\mathcal{C}$ of Γ_B. As in (2.1), we may assume that one of the following situations occurs:

(1) $0 \to \tau_B^{m+1} X \to \bigoplus_{i=1}^{s} \tau_B^m Y_i \to \tau_B^m X \to 0$ is well-defined for all $m \geq 0$;

(2) Y_s is projective and injective and $0 \to \tau_B^{m+1} X \to \bigoplus_{i=1}^{s-1} \tau_B^m Y_i \to \tau_B^m X \to 0$ is well-defined for $m \geq 1$.

Assume that case (1) holds.

Consider P' the direct sum of all indecomposable projective modules in $\mathcal{C}$. Let $P' = Be'$ and $C = B/Be'B$ which is a quotient of A by the two-sided ideal generated by an idempotent element. Observe that for $m \geq 1$, $\tau_B^m X$ is a C-module. Indeed, otherwise an indecomposable direct summand P of P' would be a predecessor of $\tau_B^m X$, for some $m \geq 1$, and hence a predecessor of $\tau_B X$, a contradiction to (1.6).

Moreover, C is a tilted algebra. Indeed, consider $\mathcal{S}$ the slice in $\mathcal{C}$ formed by those $Z \in \mathcal{C}$ such that there is a sectional path from Z to X (see [18]). We claim that any indecomposable projective P in $\mathcal{C}$ belongs to $\mathcal{S}$. Indeed, P is a predecessor of some Y_j $(1 \leq j \leq s)$ because $\bigoplus_{i=1}^{s} Y_i$ is a sincere B-module. Consider a path $P = Z_0 \to Z_1 \to \cdots \to Z_t = Y_j \to X$ and assume it is not sectional. Hence for some $1 \leq i \leq t-1$, $\tau_B Z_{i+1} = Z_{i-1}$. We may assume that the modules Z_i, $1 \leq i \leq t$, are not projectives. Then we get a path

$$P \to Z_1 \to \cdots \to Z_{i-1} \to \tau_B Z_{i+2} \to \tau_B Z_{i+3} \to \cdots \to \tau_B Y_j \to \tau_B X,$$

a contradiction. Therefore, $\tau_B \mathcal{S}$ is a slice in Γ_C and C is a tilted algebra.

We shall consider $N = \tau_B X$ as C-module. Observe that $\tau_C^m N = \tau_B^{m+1} X$ is well-defined. Consider the Grothendieck group $K_0(C)$ of C and the Coxeter matrix ϕ_C as a linear transformation $\phi_C \colon K_0(C) \to K_0(C)$. We want to show that for $m \geq 0$, the class $[\tau_C^m N] \in K_0(C)$ satisfies $[\tau_C^m N] = \phi_C^m [N]$. By [18], it is enough to show that $p\dim_C \tau_C^m N \leq 1$ and $\operatorname{Hom}_C(\tau_C^m N, {}_C C) = 0$ for $m \geq 0$.

Indeed, if $p\dim_C \tau_C^m N > 1$, there is a map $0 \neq g \in \operatorname{Hom}_C(I, \tau_C^{m+1} N)$ for some indecomposable injective C-module I. As B-module, I is also injective (otherwise, $\tau_B^- I$ is a predecessor of $\tau_C^m N = \tau_B^{m+1} X$ and $\tau_B^- I \in \operatorname{mod}_C$). Hence there is a map $0 \neq g' \in \operatorname{Hom}_B(Y_j, I)$ for some $1 \leq j \leq s$, because $\bigoplus_{i=1}^{s} Y_i$ is sincere. We get a cycle,

$$Y_j \xrightarrow{g'} I \xrightarrow{g} \tau_C^{m+1} N = \tau_B^{m+2} X \to \cdots \to \tau_B X \to Y_j,$$

a contradiction. Now, assume $0 \neq g \in \operatorname{Hom}_C(\tau_C^m N, \bar{P})$ for some indecomposable projective C-module $\bar{P}$. Since $\tau_B \mathcal{S}$ is a slice in Γ_C, then $\bar{P}$ is a predecessor of N and as B-module, $\bar{P} \in \mathcal{C}$ (use convexity of $\mathcal{C}$, see [15]). Therefore $\bar{P}$ is also projective as B-module (otherwise $\tau_B \bar{P}$ is defined and $\operatorname{Hom}_B(P', \tau_B \bar{P}) = 0$, for any projective P' in $\mathcal{S}$; hence $\tau_B \bar{P} \in \operatorname{mod}_C$, a contradiction). This contradicts (1.6).

Let Δ be a quiver such that $C = \operatorname{End}_{k\Delta}(T)$ for a tilting module T. Consider the isometry $\sigma \colon K_0(k\Delta) \to K_0(C)$, $[L] \mapsto [\operatorname{Hom}_{k\Delta}(T, L)] - [\operatorname{Ext}^1_{k\Delta}(T, L)]$, which satisfies $\phi_C \sigma = \sigma \phi_\Delta$. Observe that T does not have preinjective direct summands (since there are no projective modules in the connecting component of Γ_C) and since N is a predecessor of a slice in Γ_C, then $N = \Sigma L$ for some preinjective $k\Delta$-module L, with $\Sigma = \operatorname{Hom}_{k\Delta}(T, -)$. Hence [18] implies that $\tau_C^m N = \Sigma \tau_{k\Delta}^m L$ and $[\tau_C^m N] = \sigma \phi_\Delta^m [L]$, for $m \geq 0$.

We have the following facts:

(i) $\lim_{m\to\infty} \sqrt[m]{\dim_k \tau_C^m N}$ exists and equals $\lim_{m\to\infty} \sqrt[m]{\dim_k \tau_{k\Delta}^m L}$ (see [12] for a detailed proof).

(ii) $\lim_{m\to\infty} \sqrt[m]{\dim_k \tau_{k\Delta}^m L} = \rho(\phi_\Delta)$ (see [5], [15]).

(iii) Δ contains a subquiver $\Delta(s_1, \dots, s_m)$ with $s = \sum_{i=1}^{m} s_i$ (notation as in 2.1).

By (2.2), $1 = \lim_{m\to\infty} \sqrt[m]{\dim_k \tau_C^m N} = \rho(\phi_\Delta) \geq \rho(\phi_{\Delta(s_1,\dots,s_m)}) \geq \frac{s-2+\sqrt{s^2-4s}}{2}$. Therefore $s \leq 4$, contradicting our assumption on $s(X)$.

We get that situation (2) holds, that is, Y_s is projective and injective and for all $m \geq 0$, $0 \to \tau_B^{m+1} X \to \bigoplus_{i=1}^{s-1} \tau_B^m Y_i \to \tau_B^m X \to 0$ is well defined. Proceeding as above we get that $s - 1 \leq 4$. Hence $s(X) = 5$ and Y_s is projective and injective.

(b): Assume that $s(X) \geq 3$ and that $Y_1 \cong Y_2$. Then by (1.5), we may assume that either case (1) or case (2) in part (a) of the proof holds. In case (1), the argument above yields $1 \geq \rho(\phi_{\Delta(s_1,\dots,s_m)})$, with $\sum_{i=1}^m s_i = s(X)$ and $s_1 \geq 2$, which is impossible since $\rho(\phi_{\Delta(2,1)})$ and $\rho(\phi_{\Delta(3)})$ are both strictly bigger than 1. Hence, case (2) holds and $2 = s_1 = s(X) - 1$. □

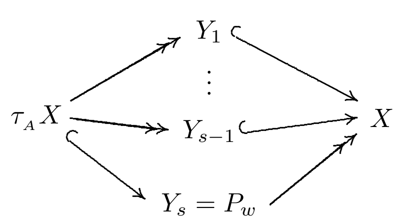
2.4. *Proof of Theorem 3.* Assume A is a cycle-finite algebra, hence tame. Let X be an indecomposable non-projective module and $\eta\colon 0 \to \tau_A X \to \bigoplus_{i=1}^{s(X)} Y_i \to X \to 0$ an almost split sequence. Assume that $s = s(X) \geq 5$. Let $B = B(\eta)$.

If X is not directing, then X belongs to a finite cycle in mod_A. By (1.6), we should have $s \leq 4$. Then X and $\tau_A X$ are directing. In case all Y_i $(1 \leq i \leq s)$ are directing, Theorem 2 shows that $s = 5$.

We may assume that Y_1 is not directing. By (1.4), there is a sectional path $\tau_A X = Z_0 \to Z_1 \to \cdots \to Z_s$ in Γ_B with Z_s an injective B-module. By (1.6), $\tau_A X$ has no projective predecessor in Γ_B.

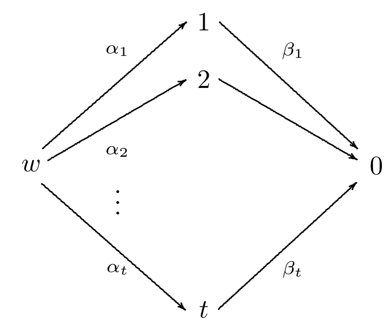
Assume that no Y_i is a projective B-module, $1 \leq i \leq s$. Then $0 \to \tau_B^{m+1} X \to \bigoplus_{i=1}^{s} \tau_B^m Y_i \to \tau_B^m X \to 0$ is well-defined for all $m \geq 0$. Consider the sequence $\tau\eta\colon 0 \to \tau_B^2 X \to \bigoplus_{i=1}^{s} \tau_B Y_i \to \tau_B X \to 0$. If all $\tau_B Y_i$ are directing as B-modules $(1 \leq i \leq s)$, then by Theorem 2 we are done. Hence by (1.4), we consider the quotient algebra $B_1 = B(\tau\eta)$ and we get a sectional path $Z'_t \to \cdots \to Z'_1 \to Z'_0 = \tau_B X$ in Γ_{B_1} with Z'_t a projective B_1-module. Since Z'_t cannot be a projective B-module predecessor of $\tau_A X$, then B_1 is a proper quotient of B. Consider the sequence $\tau^2\eta\colon 0 \to \tau_B^3 X \to \bigoplus_{i=1}^{s} \tau_B^2 Y_i \to \tau_B^2 X \to 0$. Again either all $\tau_B^2 Y_i$ are directing as B_1-modules and $s \leq 5$ or $B_2 = B(\tau^2\eta)$ is a proper quotient of B_1. Repeating the process finitely many steps we end with an algebra $B_t = B(\tau^t\eta)$ where all $\tau_B^t Y_i$ are directing. Theorem 2 yields $s = 5$ and some $\tau_B^t Y_j$ is projective $(1 \leq j \leq s)$, a contradiction.

We may therefore assume that Y_s is a projective B-module. By the dual argument, we may suppose that some Y_j $(1 \leq j \leq s)$ is an injective B-module. As in (2.1), $j = s$. Let $Y_s = P_w$ be a projective-injective B-module. Then the irreducible maps of η look as follows:

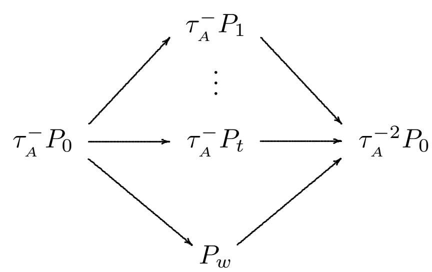


Since $\tau_A X$ is directing in mod_A, then $\tau_A X(w) = 0$. Hence $Y_i(w) = 0$ for $1 \leq i \leq s-1$. Then $Y_s = P_w$ is a sincere B-module and $B = C[M] = \begin{pmatrix} C & M \\ 0 & k \end{pmatrix}$ is a one-point extension of an algebra C by the module $M = \operatorname{rad} P_w$. Since P_w is injective, also $M = \tau_A X$. Hence $\operatorname{rad} P_w$ is directing as A-module, and therefore P_w is directing as B-module. We get that B is a tilted algebra and by Theorem 2 we are done. □

2.5 Examples. a) The bound in Theorems 1 and 2 is optimal. Consider A_t the algebra given by the quiver Q

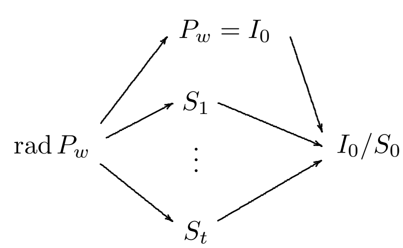


and ideal I_t generated by $\sum_{i=1}^{t} \beta_i\alpha_i$. Then $\tilde{\rho}(A_t) = \frac{t-2+\sqrt{t^2-4}}{2}$. There is an almost split sequence as follows:



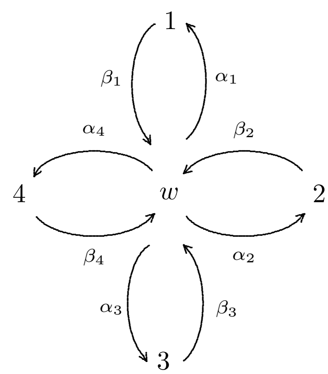
that is, $s(\tau_A^{-2}P_0) = t+1$. The algebra A_t is tame if and only if $t \leq 3$.

b) Consider B_t the algebra given by the above quiver Q and ideal I'_t generated by $\beta_i\alpha_i - \beta_1\alpha_1$, $i = 2, \dots, t$. Again $\tilde{\rho}(B_t) = \frac{t-2+\sqrt{t^2-4}}{2}$. There is an almost split sequence as follows:

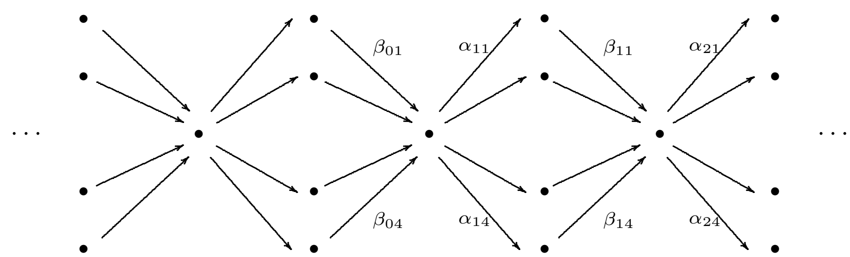


That is, $s(I_0/S_0) = t+1$. The algebra B_t is tame for $t \leq 4$.

c) Consider the algebra C given by the quiver

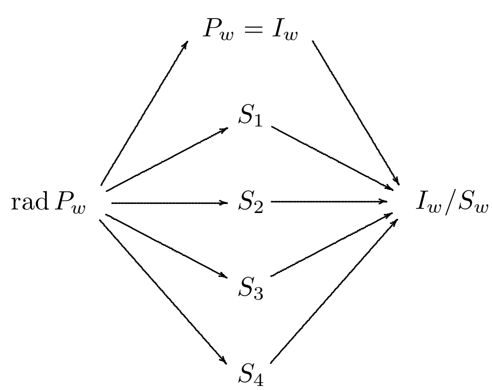


with ideal I generated by $\beta_i\alpha_i - \beta_1\alpha_1$, $i = 2,3,4$ and $\alpha_j\beta_i = 0$ for any $1 \le i,j \le 4$. Then C has a Galois covering $\tilde{C}$ given by the quiver $\tilde{Q}$



and ideal $\tilde{I}$ generated by $\beta_{ji}\alpha_{ji} - \beta_{j1}\alpha_{j1}$ for $i = 2,3,4$ and $j \in \mathbb{Z}$ and $\alpha_{j+1,s}\beta_{j,t} = 0$ for all $1 \le s,t \le 4$ and $j \in \mathbb{Z}$.

It is easy to check that $\tilde{C}$ has no hypercritical or *pg*-critical subcategories and hence $\tilde{C}$ is tame of polynomial growth [19]. Since the covering $\pi\colon \tilde{C} \to C$ is defined by the action of the free group $\mathbb{Z}$, then C is tame [13], [19]. There is an almost split sequence in mod_C of the form



and there are cycles in mod_C as follows:

$$S_i \to I_i \to I_w = P_w \to P_i \to S_i$$

for all $i = 1, \dots, 4$.

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